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Discrete Mathematics 306 (2006) 199–209

DISCRETE MATHEMATICS

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On finite matroids with two more hyperplanes than points

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Received 24 March 2003; received in revised form 2 November 2004; accepted 10 October 2005
Available online 23 January 2006

Abstract

One of the most interesting results about finite matroids of finite rank and generalized projective spaces is the result of Basterfield, Kelly and Green (1968/1970) (J.G. Basterfield, L.M. Kelly, A characterization of sets of n points which determine n hyperplanes, in: Proceedings of the Cambridge Philosophical Society, vol. 64, 1968, pp. 585–588; C. Greene, A rank inequality for finite geometric lattices, J. Combin Theory 9 (1970) 357–364) affirming that any matroid contains at least as many hyperplanes as points, with equality in the case of generalized projective spaces. Consequently, the goal is to characterize and classify all matroids containing more hyperplanes than points. In 1996, I obtained the classification of all finite matroids containing one more hyperplane than points. In this paper a complete classification of finite matroids with two more hyperplanes than points is obtained. Moreover, a partial contribution to the classification of those matroids containing a certain number of hyperplanes more than points is presented.

Keywords: Linear spaces; Matroids; Projective spaces

1. Introduction

A linear space is an incidence structure $\mathbb{L} = (\mathscr{P}, \mathscr{L})$ consisting of a non-empty set $\mathscr{P}$, whose elements are called points, and a family $\mathscr{L}$ of subsets of $\mathscr{P}$, called lines, such that any two distinct points belong to a unique line, any line contains at least two points and there are at least three points not on the same line. A subspace of a linear space $\mathbb{L}$ is a subset X of points containing the lines joining any pair of its distinct points. Clearly, every intersection of subspaces is a subspace, thus it is possible to define the closure of a subset T of points as the intersection $[T]$ of all subspaces of $\mathbb{L}$ containing T. Moreover, for every pair of subsets X and Y of $\mathscr{P}$, let $X \vee Y$ be the closure $[X \cup Y]$.

According to Buekenhout (see [4]), an n-dimensional linear space is a linear space $\mathbb{L}_n = (\mathscr{P}, \mathscr{L})$ containing $n+2$ disjoint families of subspaces $\mathscr{B}_i$, $i = -1, \dots, n$, whose elements are called i-subspaces (or subspaces of dimension i), satisfying the following properties:

(i) $\mathscr{B}_{-1} := \{\emptyset\}$, $\mathscr{B}_0 := \mathscr{P}$, $\mathscr{B}_1 := \mathscr{L}$, $\mathscr{B}_n := \{\mathscr{P}\}$.
(ii) If V is an i-subspace ($i \leqslant n-1$) and p is a point not on V, then there exists a unique $(i+1)$-subspace containing p and V.
(iii) If a j-subspace W contains an i-subspace V, then $i \leqslant j$.
(iv) The intersection of subspaces of $\mathscr{B}_{-1} \cup \cdots \cup \mathscr{B}_n$ is still a subspace.

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0012-365X/$ - see front matter
doi:10.1016/j.disc.2005.10.023

From property (i), points and lines of $\mathbb{L}_n$ are subspaces of dimension 0 and 1, respectively; moreover, for $i=2, n-1$, the i-subspaces are also called planes and hyperplanes, respectively. A finite n-dimensional linear space is an n-dimensional linear space with a finite set of points. The number of i-subspaces of a finite n-dimensional linear space will be denoted by W_i. As usual, we put $W_0 := v$.

Using some elementary facts of the theory of finite matroids, (see [2,15]), it is quite easy to see that the i-subspaces of a finite n-dimensional linear space are the flats of rank $i+1$ of a simple matroid of rank $n+1$, and viceversa. Furthermore, the flats of a simple matroid ordered by inclusion define a geometric lattice and, by a fundamental result of Birkhoff (see [2]), the correspondence mapping every simple matroid onto the lattice of its subspaces is a bijection, thus any finite n-dimensional linear space is either a simple matroid of rank $n+1$, or a geometric lattice of height $n+1$.

The sum of an n-dimensional linear space $\mathbb{L}=(\mathscr{P}, \mathscr{L})$ and an n'-dimensional linear space $\mathbb{L}'=(\mathscr{P}', \mathscr{L}')$ is the linear space $\mathbb{L} \oplus \mathbb{L}'$ whose points are those of $\mathscr{P}$ and $\mathscr{P}'$ and whose lines are the elements of $\mathscr{L} \cup \mathscr{L}'$ and all the 2-sets $\{x, y\}$, with $x \in \mathscr{P}$ and $y \in \mathscr{P}'$. Clearly, the sum $\mathbb{L} \oplus \mathbb{L}'$ is a $(n+n'+1)$-dimensional linear space.

The residue of an n-dimensional linear space $\mathbb{L}=(\mathscr{P}, \mathscr{L})$ at a point x of $\mathscr{P}$ is the $(n-1)$-dimensional linear space $\mathbb{L}_x$ whose i-subspaces are the $(i+1)$-subspaces of $\mathbb{L}$ passing through x.

A generalized projective space is a linear space satisfying the Veblen–Young axiom: any line intersecting two sides of a triangle intersects the third side, too. It is well known that every finite generalized projective space is the sum of projective spaces.

A typical problem for finite n-dimensional linear spaces is to derive geometric properties from algebraic relations among arithmetical parameters of the space. The first results in this sense are the so-called Fundamental Theorem of de Bruijn-Erdös-Hanani (see [5,11]) and its extensions to higher dimensional finite linear spaces, due to Basterfield and Kelly and Greene (see [1,10]). These results are summarized in the following theorem.

Theorem 1.1. *Let $\mathbb{L}$ be a finite n-dimensional linear space on v points. Then $W_i \geqslant v$, for every $i=1, \ldots, n-1$. Moreover, equality holds for some dimension i if, and only if, $i=n-1$ and $\mathbb{L}$ is an n-dimensional generalized projective space.*

In the papers [3,8,9], we find the complete classification of finite n-dimensional linear spaces satisfying the equality $W_{n-1}=v+1$. We call these spaces Bridges spaces, since the two-dimensional case has been solved by Bridges in 1972. Precisely, we have the following theorem.

Theorem 1.2. *Let $\mathbb{L}$ be an n-dimensional Bridges space. Then one of the following cases hold*:

(i) *(Bridges [3]) $n=2$ and $\mathbb{L}$ is either a projective plane with one point deleted, or the Fano plane with two points deleted*;
(ii) *(Ferrara Dentice [8,9]) $n \geqslant 3$ and $\mathbb{L}$ is either a Galois projective space $PG(n, q)$ with one point deleted or the sum of a d-dimensional generalized projective space and a d'-dimensional Bridges space, for $d+d'=n-1$.*

The purpose of this paper is to study finite n-dimensional linear spaces satisfying the equality $W_{n-1}=v+2$. According to the previous case, we call these spaces *de Witte spaces*, since the two-dimensional case has been solved by de Witte in 1976. Precisely, the following theorem holds.

Theorem 1.3 (*deWitte [7]*). *Let $\mathbb{L}$ be a finite linear space containing two more lines than points. Then $\mathbb{L}$ is one of the following structures*:

(i) *a finite projective plane π_q of order $q \geqslant 3$, with two points deleted*;
(ii) *the finite projective plane π_3 of order 3 with three collinear points deleted*;
(iii) *the Fano quasi-plane, i.e. the finite linear space obtainable from the Fano plane $PG(2, 2)$ by "breaking up" one line into three*;
(iv) *the Lin's cross, i.e. the finite linear space on 6 points with two intersecting lines of length 4 and 3, respectively*;
(v) *the affine plane $AG(2, 2)$.*

In 1996, De Vito and Lo Re (see [6]) classified all three-dimensional de Witte spaces, proving that a three-dimensional de Witte space is either a projective space $PG(3, q)$ with two points deleted, or the sum of a point p and one of the two-dimensional de Witte spaces of (i)–(v) of Theorem 1.3 above.

The main result of this paper is to prove that the classification of three-dimensional de Witte spaces can be extended to any dimension n. More precisely, I prove the following theorem.

Theorem 1.4. *An n-dimensional de Witte space* $\mathbb{L}$, $n \geqslant 4$, *is one of the following structures*:

(i) *a projective space* $PG(n, q)$ *of order* $q \geqslant 3$, *with two points deleted*;
(ii) *the sum of a d-dimensional generalized projective space and a* d'*-dimensional de Witte space, for* $d + d' = n - 1$;
(iii) *the sum of two Bridges spaces of dimensions* d *and* d', *for* $d + d' = n - 1$.

In 1976 Totten [14] classified all restricted linear spaces, (i.e. linear spaces on v points and b lines such that $b = v + \alpha$ and $\alpha^2 \leqslant v$). Later on, Metsch improved the theorem of Totten, by determining all linear spaces with $\alpha^2 \leqslant b$ (see [12, Theorem 8.6]) and dealing with the theorem of Totten for three-dimensional linear spaces (Metsch [13]). In this context, the following question naturally arises: what happens in n-dimensional linear spaces on v points containing α hyperplanes more than points? (I define these spaces Totten spaces of parameters (v, α)). Unfortunately, the techniques used for the proof of Theorem 1.4 do not work for a general α. Anyway, a partial answer to the classification problem for n-dimensional Totten spaces can be obtained. The following result is proved in Section 2.

Theorem 1.5. *If an n-dimensional Totten space* $\mathbb{L}$ *of parameters* (v, α), $\alpha \geqslant 2$ *contains* $v + \alpha - 1$ *hyperplanes with a non-empty intersection* $\mathscr{I}$, *then* $\mathscr{I}$ *is a point* x_0, *the remaining hyperplane* B_0 *is an* $(n-1)$*-dimensional Totten space of parameters* $(v - 1, \alpha)$ *and* $\mathbb{L}$ *is the sum of* x_0 *and* B_0.

Organization of the paper. Section 2 will be devoted to examine the case in which a Totten space of parameters (v, α) contains $v + \alpha - 1$ hyperplanes with a non-empty intersection. Thus, the section contains the complete proof of Theorem 1.5 and the classification of de Witte spaces containing $v + 1$ hyperplanes with a non-empty intersection. In order to complete the proof of Theorem 1.4, the remaining case of a de Witte space in which $v + 1$ hyperplanes always intersect in the empty set, will be treated in Section 3. In this case, many subcases occur and they will be tackled in four subsections.

I would like to point out that several proofs of this paper will be omitted, since either they can be easily obtained by induction, or they are similar to those of the corresponding claims in [8].

2. The classification of Totten spaces of parameters (v, α) containing $v + \alpha - 1$ distinct hyperplanes with a non-empty intersection

Let $\mathbb{L} = (\mathscr{P}, \mathscr{L})$ be a finite n-dimensional Totten space on v points and let $B_1, B_2, \dots, B_{v+\alpha-1}$ be $v + \alpha - 1$ distinct hyperplanes of $\mathbb{L}$ with a non-empty intersection. Moreover, let $\mathscr{I}$ be the subspace $B_1 \cap \dots \cap B_{v+\alpha-1}$ and B_0 be the remaining hyperplane of $\mathbb{L}$. The following claims are very easy to prove (besides, the proofs are very similar to those of Section 2 of [8]).

(i) The subspace $\mathscr{I}$ has dimension at most $n - 3$.
(ii) The subspace $\mathscr{I}$ is not contained in B_0.
(iii) If $\mathscr{I}$ is at least a line, then every line contained in $\mathscr{I}$ meets B_0.
(iv) Every point of $\mathscr{I} \backslash B_0$ is on exactly $v - 1$ lines of $\mathbb{L}$, and any such line contains exactly two points. It follows that $\mathscr{I} \backslash B_0$ is a point x_0 and every line through x_0 meets B_0.
(v) B_0 contains exactly $v + \alpha - 1$ hyperplanes.

From (iv) and (v), Theorem 1.5 is proved and also the classification of de Witte spaces containing $v + 1$ hyperplanes with a non-empty intersection is obtained.

3. The case of a de Witte space in which $v + 1$ hyperplanes always intersect in the empty set

Let $B_0, B_1, \dots, B_v, B_{v+1}$ be the hyperplanes of the n-dimensional de Witte space $\mathbb{L} = (\mathscr{P}, \mathscr{L})$, and denote by B'_i the complement $\mathscr{P} \backslash B_i$, for every $i = 0, \dots, v + 1$. A set $\{x_1, \dots, x_k\}$ of points of $\mathscr{P}$ is called a transversal of $\{B'_{i_1}, \dots, B'_{i_k}\}$ if $x_j \in B'_{i_j}$ and $x_j \neq x_t$, for every $j, t = 1, \dots, k$, with $j \neq t$.

The same arguments used in the proof of Proposition 3.1 of [8] allow us to obtain the following result.

Proposition 3.1. *There exist v hyperplanes whose complements have a transversal.*

For every point x of $\mathbb{L}$, let $W_i(x)$ be the number of i-subspaces of $\mathbb{L}$ passing through x, for every $i = 1, \ldots, n-1$. The next proposition easily follows from Theorem 1.1 and provides a lower bound for $W_{n-1}(x)$.

Proposition 3.2. *For every hyperplane B of $\mathbb{L}$ and for every point x not on B, the inequality $W_{n-1}(x) \geqslant |B|$ holds. Moreover, if equality holds, then B is a generalized projective space of dimension $n-1$.*

By Proposition 3.1, let $\{B_1, B_2, \ldots, B_v\}$ be a set of v hyperplanes of $\mathbb{L}$ whose complements have a transversal $\{x_1, x_2, \ldots, x_v\}$. Since $x_i \notin B_i$, from Proposition 3.2, it follows that $W_{n-1}(x_i) \geqslant |B_i|$ for every $i = 1, \ldots, v$.

Let B_0 and B_{v+1} be the remaining two hyperplanes of $\mathbb{L}$. The principle of double-counting on the incident pairs (point, hyperplane) of $\mathbb{L}$ gives

$$\sum_{x \in \mathscr{P}} W_{n-1}(x) = \sum_{B \in \mathscr{B}_{n-1}} |B|$$

and hence the following equality holds:

$$|B_0| + |B_{v+1}| = \sum_{i=1}^{v} (W_{n-1}(x_i) - |B_i|). \tag{*}$$

Proposition 3.3. *There exist at least two distinct indices $i, j \in \{1, \ldots, v\}$ such that $|B_i| \leqslant W_{n-1}(x_i) \leqslant |B_i| + 1$ and $|B_j| \leqslant W_{n-1}(x_j) \leqslant |B_j| + 1$.*

Proof. By Proposition 3.2, the inequality $|B_i| \leqslant W_{n-1}(x_i)$ holds for every $i = 1, \ldots, v$.

If $W_{n-1}(x_i) \geqslant |B_i| + 2$ for every $i = 1, \ldots, v$, then, by Eq. $(*)$ above, we have

$$2v \leqslant \sum_{i=1}^{v} (W_{n-1}(x_i) - |B_i|) = |B_0| + |B_{v+1}| \leqslant v + |B_0 \cap B_{v+1}|,$$

and hence $|B_0 \cap B_{v+1}| \geqslant v = |\mathscr{P}|$, a contradiction.

It follows that there exists at least an index $i \in \{1, \ldots, v\}$ such that $W_{n-1}(x_i) \leqslant |B_i| + 1$.

If there exists exactly one index i such that $W_{n-1}(x_i) \leqslant |B_i| + 1$, then either $W_{n-1}(x_i) = |B_i| + 1$ or $W_{n-1}(x_i) = |B_i|$, and, for every $j \in \{1, \ldots, v\} \backslash \{i\}$, $W_{n-1}(x_j) \geqslant |B_j| + 2$.

If $W_{n-1}(x_i) = |B_i| + 1$, from condition $(*)$ we have $|B_0 \cap B_{v+1}| \geqslant v - 1$, a contradiction.

Hence we have that $W_{n-1}(x_i) = |B_i|$ and, from condition $(*)$ again, $|B_0 \cap B_{v+1}| \geqslant v - 2$. It follows that $|B_0 \cap B_{v+1}| = v - 2$ and $\mathscr{P}$ contains exactly two points $x \in B_0 \backslash B_{v+1}$ and $y \in B_{v+1} \backslash B_0$ such that $\mathscr{P} = \{x, y\} \cup (B_0 \cap B_{v+1})$. Every hyperplane $B_1, \ldots, B_v$ does not contain $C := B_0 \cap B_{v+1}$, otherwise it coincides either with B_0 or with B_{v+1}, a contradiction. It follows that every hyperplane $B_0, B_1, \ldots, B_v$ necessarily contains both x and y. Thus, we have that the $v + 1$ hyperplanes $\{B_0, B_1, \ldots, B_v\}$ of $\mathbb{L}$ intersect at the point x, a contradiction. $\square$

Let h be the number of indices $i \in \{1, \ldots, v\}$ such that $|B_i| \leqslant W_{n-1}(x_i) \leqslant |B_i| + 1$. By the previous proposition, $h \geqslant 2$ and, for the sake of simplicity, we can suppose that the h pairs (x_i, B_i) such that $|B_i| \leqslant W_{n-1}(x_i) \leqslant |B_i| + 1$ are the first h. Since the hyperplanes of $\mathbb{L}$ passing through x_i are at least as many as the hyperplanes of B_i, if s_i is the number of hyperplanes of B_i, for every $i \in \{1, \ldots, h\}$, one, and only one, of the following conditions holds:

(i) $|B_i| = s_i = W_{n-1}(x_i)$;
(ii) $|B_i| = s_i$, $W_{n-1}(x_i) = |B_i| + 1$;
(iii) $s_i = W_{n-1}(x_i) = |B_i| + 1$.

In the following three subsections, I examine the three cases above, with only two exceptions, which are treated in subsection 3.4.

3.1. The case (i)*, where one of the hyperplanes $B_1, \ldots, B_h$ satisfies the equalities $|B_i| = s_i = W_{n-1}(x_i)$*

For the sake of simplicity, let us suppose that equality (i) is satisfied by the hyperplane B_1. First of all we observe that, by Theorem 1.1, B_1 is a generalized projective space of dimension $n-1$. We have the following results.

Proposition 3.4. *For every $j = 1, \ldots, n-1$, any j-subspace X of $\mathbb{L}$ passing through x_1 intersects B_1 at a $(j-1)$-dimensional subspace.*

Proof. The proof easily proceeds by induction on $k = n - 1 - j$, $0 \leqslant k \leqslant n-2$, the first step being a consequence of the equality $s_1 = W_{n-1}(x_1)$. $\square$

Proposition 3.5. *If B_1 is the sum of two generalized projective subspaces, then $\mathbb{L}$ is the sum of a d-dimensional generalized projective space and a d'-dimensional de Witte space, for $d + d' = n - 1$.*

Proof. Assume $B_1 = X \oplus Y$, where X and Y are generalized projective spaces of dimension h and k, respectively, with $h+k=n-2$, and put $V = x_1 \vee X$ and $V' = x_1 \vee Y$. By Proposition 3.4, every line passing through x_1 intersects B_1, hence it is contained either in V or in V', and $\mathscr{P} = V \cup V'$. Moreover, $V \cap V' = \{x_1\}$, otherwise the line joining x_1 to a point y of $V \cap V' \backslash \{x_1\}$ intersects B_1 (by Proposition 3.4) and so $(x_1 \vee y) \cap B_1 \subseteq (V \cap V') \cap B_1 = X \cap Y = \emptyset$, a contradiction. It follows that $v = |\mathscr{P}| = |V| + |V'| - 1$. Now, it is easy to see that the hyperplanes of $\mathbb{L}$ are obtained by joining either an hyperplane of X to V', or an hyperplane of Y to V, or an hyperplane of V not through x_1 to an hyperplane of V' not through x_1. Let W_T be the number of the hyperplanes of a subspace T of $\mathbb{L}$. We have $W_{n-1} = W_X + W_Y + (W_V - W_X)(W_{V'} - W_Y)$. Since $W_{n-1} = v + 2$, it follows that

$$|V| + |V'| + 1 = W_X + W_Y + (W_V - W_X)(W_{V'} - W_Y). \tag{3.5.1}$$

By Theorem 1.1, $W_V \geqslant |V|$. Now, we proceed in several steps.

Step 1: If $W_V \geqslant |V| + 2$, then $W_V = |V| + 2$ and $\mathbb{L}$ is the sum of the generalized projective space Y and the de Witte space V.

Since $|V| \geqslant |X| + 1 = W_X + 1$, from equality (3.5.1) it follows that

$$1 - W_{V'} + W_Y \geqslant \frac{(W_{V'} - |V'|) + (W_{V'} - (W_Y + 1))}{|V| - W_X}. \tag{3.5.2}$$

Moreover, $W_{V'} \geqslant |V'|$ by Theorem 1.1, and $W_{V'} \geqslant W_Y + 1$, since for every hyperplane D of Y, $x_1 \vee D$ is an hyperplane of V', thus, from (3.5.2) it follows that $1 - W_{V'} + W_Y \geqslant 0$, and so $W_{V'} = W_Y + 1$. From (3.5.2) again, we have that $W_{V'} = |V'|$. Hence V' is a generalized projective space and, from $|V'| = W_{V'} = W_Y + 1 = |Y| + 1$, it follows that $V' = x_1 \oplus Y$. From (3.5.1), the equalities $W_{V'} = W_Y + 1$ and $W_Y = |V'| - 1$ imply that $W_V = |V| + 2$, hence V is a $(h+1)$-dimensional de Witte space and $\mathbb{L} = Y \oplus V$.

Obviously, the above arguments work also for V', thus Step 2 is trivial.

Step 2: If $W_{V'} \geqslant |V'| + 2$, then $W_{V'} = |V'| + 2$ and $\mathbb{L}$ is the sum of the generalized projective space X and the de Witte space V'.

From now on, we can suppose that $W_V \leqslant |V| + 1$ and, equivalently, $W_{V'} \leqslant |V'| + 1$. Clearly, this leads to the following four possibilities:

1. $W_V = |V|$ and $W_{V'} = |V'| + 1$;
2. $W_V = |V| + 1$ and $W_{V'} = |V'|$;
3. $W_V = |V| + 1$ and $W_{V'} = |V'| + 1$;
4. $W_V = |V|$ and $W_{V'} = |V'|$.

Step 3: Cases 1 and 2 do not occur.

We can interchange the role of V and V', hence only case 1 needs to be considered. If $W_V = |V|$ and $W_{V'} = |V'| + 1$, from (3.5.1) we have $(|V| - |X| - 1)(|V'| - |Y|) = 1$, hence $|V| - |X| - 1 = |V'| - |Y| = 1$. In this case we have a contradiction, since $V' = x_1 \oplus Y$, as it contains exactly one more point than its hyperplane Y, but the sum of a point and a generalized projective space is a generalized projective space too, while V' is a Bridges space.

Step 4: Case 3 does not occur, too.

If $W_V = |V| + 1$ and $W_{V'} = |V'| + 1$, then, from (3.5.1) again, we have $(|V| - |X|)(|V'| - |Y|) = 0$, a contradiction, since $|V| \geqslant |X| + 1$ and $|V'| \geqslant |Y| + 1$.

Step 5: If $W_V = |V|$ and $W_{V'} = |V'|$, then $\mathbb{L}$ is the sum of a Lin's cross and an $(n-3)$-dimensional generalized projective space.

From (3.5.1) we have $(|V| - |X|)(|V'| - |Y| - 1) = |V'| - |Y| + 1$, hence either $|V| - |X| = 2$ and $|V'| - |Y| = 3$, or $|V| - |X| = 3$ and $|V'| - |Y| = 2$. As the arguments also work if we interchange V and V', as well as X and Y, we can consider only the first case. Therefore, let $V \backslash X = \{x_1, x\}$ and $V' \backslash Y = \{x_1, y, z\}$. The line $x_1 \vee x$ of the generalized projective space V intersects the hyperplane X of V at a point x'. If X contains a line M with at least three points and passing through x', then the projective plane $x_1 \vee M$ of V does not satisfy the Veblen–Young axiom. It follows that every line of X through x' contains exactly two points and X is the sum of x' and an $(h-1)$-dimensional generalized projective space X'. Let now consider the subspace V'. If the points x_1, y and z were non-collinear, then the projective plane π passing through them did not satisfy the Veblen–Young axiom. Hence x_1, y and z are three collinear points of V', the line of V' passing through them intersects Y at a point y' and every line of Y passing through y' contains exactly two points. It follows that Y is the sum of the point y' and a $(k-1)$-dimensional generalized projective space Y'. Finally, it is clear that the linear space $\mathbb{L}'$ whose point-set is $\{x_1, x, x', y, z, y'\}$ is the Lin's cross, and $\mathbb{L} = (X' \oplus Y') \oplus \mathbb{L}'$ is the sum of the $(n-3)$-dimensional generalized projective space $X' \oplus Y'$ and the two-dimensional de Witte space $\mathbb{L}'$. □

Remark 3.6. By Proposition 3.5, from now on we can suppose that for every $i = 1, \ldots, h$, every hyperplane B_i satisfying the equalities $|B_i| = s_i = W_{n-1}(x_i)$ is an $(n-1)$-dimensional projective space $PG(n-1, q_i)$ of order $q_i \geqslant 2$. This occurrence will be treated in subsection 3.4.

3.2. The case (ii), *where one of the hyperplanes $B_1, \ldots, B_h$ satisfies the equalities $|B_i| = s_i$ and $W_{n-1}(x_i) = |B_i| + 1$*

Suppose that one of the hyperplanes B_i of $\mathbb{L}$, say B_1, satisfies the equalities $|B_1| = s_1$, $W_{n-1}(x_1) = |B_1| + 1$. First of all, the following results hold.

Proposition 3.7. (i) *There exists a unique hyperplane B^* of $\mathbb{L}$ passing through x_1 and intersecting B_1 at a subspace C of dimension at most $n-3$.*

(ii) *There exists a unique line L passing through x_1 and missing B_1. Moreover, the line L is contained in $B^* \backslash (x_1 \vee C)$.*

(iii) *The residue $\mathbb{L}_{x_1}$ of $\mathbb{L}$ at the point x_1 is a generalized projective space of dimension $n-1$ containing $|B_1| + 1$ points.*

(iv) *The subspace C has dimension $n-3$.*

Proof. (i) It easily follows from equalities $W_{n-1}(x_1) = |B_1| + 1 = s_1 + 1$.

(ii) As C has dimension at most $n-3$, the subspace $x_1 \vee C$ has dimension at most $n-2$, thus B^* contains a point $y \notin x_1 \vee C$. Clearly, the line $L = x_1 \vee y$ does not intersect B_1. Moreover, L is the unique line passing through x_1 and missing B_1. Indeed, from Theorem 1.1 on the residue $\mathbb{L}_{x_1}$ we have $W_{n-1}(x_1) \geqslant W_1(x_1)$, thus $W_{n-1}(x_1) \geqslant W_1(x_1) \geqslant |B_1| + 1 = W_{n-1}(x_1)$ and the equality $W_1(x_1) = |B_1| + 1$ holds.

(iii) By the previous case, the statement follows from the equality $W_{n-1}(x_1) = W_1(x_1) = |B_1| + 1$.

(iv) By contradiction, let us suppose that the subspace C has dimension at most $n-4$. Then the subspace $L \vee C$ has dimension at most $n-2$, and B^* contains a point $w \notin L \vee C$. The line $x_1 \vee w$ is different from L, thus, from (ii), $x_1 \vee w$ intersects B_1 at a point $w' \in B^* \cap B_1 = C$. It follows that $w \in L \vee C$, a contradiction. □

Conditions (i) and (iv) of Proposition 3.7 have a natural extension to j-dimensional subspaces of $\mathbb{L}$ passing through x_1, for every $j = 1, \ldots, n-1$. More precisely, the following holds.

Proposition 3.8. *For every $j = 1, \ldots, n-1$, there exists a unique j-subspace X_j of $\mathbb{L}$ passing through x_1 and intersecting B_1 at a subspace of dimension $j-2$.*

Proof. The proof easily proceeds by induction on $k = n-1-j$, $0 \leqslant k \leqslant n-2$, the first step being conditions (i) and (iv) of Proposition 3.7. □

By condition (iii) of Proposition 3.7, the residue $\mathbb{L}_{x_1}$ is an $(n-1)$-dimensional generalized projective space on $|B_1|+1$ points. Let $\mathscr{S}_i$ be the family of i-subspaces of $\mathbb{L}_{x_1}$, for every $i=-1, 0, 1, \ldots, n-1$. It is well known that $\mathscr{S}_{-1}=\{\emptyset\}$, $\mathscr{S}_i$ is the set of all $(i+1)$-subspaces of $\mathbb{L}$ passing through x_1 for every $i=0, \ldots, n-2$ and $\mathscr{S}_{n-1}=\{\mathscr{S}_0\}$. By Proposition 3.8, it is easy to see that the $n+1$ pairwise disjoint families $\mathscr{S}'_{-1}=\{\emptyset\}$, $\mathscr{S}'_i=\mathscr{S}_i\backslash\{X_{i+1}\}$ (for $i=0, \ldots, n-2$), $\mathscr{S}'_{n-1}=\{\mathscr{S}'_0\}$, define an $(n-1)$-dimensional linear space $\mathbb{L}'$ on $W_1(x_1)-1=|B_1|$ points. We have the following result.

Proposition 3.9. *The* $(n-1)$-*dimensional linear space* $\mathbb{L}'$ *is a generalized projective space isomorphic to* B_1.

Proof. Let $\varphi : B_1 \longrightarrow \mathscr{S}'_0$ be the map defined by $\varphi(x) := x_1 \vee x$. Clearly, φ is injective and, since $|B_1| = |\mathscr{S}'_0|$, φ is also bijective. Moreover, φ maps lines of B_1 onto lines of $\mathbb{L}'$, and φ^{-1} maps lines of $\mathbb{L}'$ onto lines of B_1, since every line of $\mathbb{L}'$ is a plane π of $\mathbb{L}$ passing through x_1 and intersecting B_1 at a line L, thus $\varphi^{-1}(\pi) = L$. It follows that φ is a collineation and $\mathbb{L}'$ is a generalized projective space. $\square$

By condition (iii) of Proposition 3.7, and by Proposition 3.9, we have a contradiction, since it is impossible to obtain a generalized projective space by deleting a point from a generalized projective space of the same dimension. Thus, the following theorem holds.

Proposition 3.10. *For every* $i = 1, \ldots, h$, *the case* $|B_i| = s_i$ *and* $W_{n-1}(x_i) = |B_i| + 1$ *does not occur.*

3.3. The case (iii), *where one of the hyperplanes* $B_1, \ldots, B_h$ *satisfies the equalities* $W_{n-1}(x_i) = s_i = |B_i| + 1$

For the sake of simplicity, let us suppose that the equality is satisfied by the hyperplane B_1. First of all, we observe that B_1 is a $(n-1)$-dimensional Bridges space. We have the following results.

Proposition 3.11. (i) *There exists at most one line of* $\mathbb{L}$ *passing through* x_1 *and missing* B_1.

(ii) *For every* $j = 2, \ldots, n-1$, *every j-subspace of* $\mathbb{L}$ *passing through* x_1 *intersects* B_1 *at a* $(j-1)-$ *subspace*.

(iii) *There exists exactly one line passing through* x_1 *and missing* B_1 *if*, *and only if*, B_1 *is an* $(n-1)$-*dimensional projective space with one point deleted.*

Proof. (i) The statement easily follows from the inequalities $|B_1| \leqslant W_1(x_1) \leqslant W_{n-1}(x_1) = |B_1| + 1$.

(ii) The proof easily proceeds by induction on $k = n - 1 - j$, $0 \leqslant k \leqslant n-3$, the first case being the equality $W_{n-1}(x_1) = s_1$.

(iii) From the inequalities $|B_1| \leqslant W_1(x_1) \leqslant W_{n-1}(x_1) = |B_1| + 1$, it follows that there exists exactly one line passing through x_1 and missing B_1 if, and only if, $W_1(x_1) = W_{n-1}(x_1) = |B_1| + 1$. In this case, the residue $\mathbb{L}_{x_1}$ is an $(n-1)$-dimensional generalized projective space on $|B_1| + 1$ points, and the injective map $\varphi : B_1 \longrightarrow \mathbb{L}_{x_1}$ defined by $\varphi(x) = x_1 \vee x$, for every point x of B_1, transforms lines onto lines, preserving incidence. From case (ii) above, φ is a bijection on lines, planes, up to hyperplanes, hence B_1 is a Bridges space containing exactly one less point than a generalized projective space and the same number of h-subspaces, for every $h = 1, \ldots, n-2$. It is easy to see that this happens if, and only if, B_1 is an $(n-1)$-dimensional projective space with one point deleted. $\square$

Using the classification of Bridges spaces of Theorem 1.2, in the following proposition I examine the case in which B_1 splits.

Proposition 3.12. *If* B_1 *is the sum of a generalized projective space and a Bridges space*, *then* $\mathbb{L}$ *either is the sum of a generalized projective space and a de Witte space*, *or it is the sum of two Bridges spaces of dimensions d and* d', *with* $d + d' = n - 1$.

Proof. Assume $B_1 = X \oplus Y$, where X is a generalized projective space of dimension h and Y is a Bridges space of dimension k, with $h + k = n - 2$, and put $V = x_1 \vee X$ and $V' = x_1 \vee Y$. By the case (iii) of Proposition 3.11, every line of $\mathbb{L}$ passing through x_1 intersects B_1, hence $\mathscr{P} = V \cup V'$. Moreover, $V \cap V' = \{x_1\}$, otherwise the line joining x_1 with a point y of $V \cap V'\backslash\{x_1\}$ intersects B_1, and so $(x_1 \vee y) \cap B_1 \in X \cap Y = \emptyset$, a contradiction. It follows that $v = |\mathscr{P}| = |V| + |V'| - 1$. The same arguments as in the proof of Proposition 3.5 produce the equality (3.5.1).

By Theorem 1.1, $W_V \geqslant |V|$. We proceed in several steps.

Step 1: $W_V \leqslant |V|+1$.

By contradiction, assume that $W_V \geqslant |V|+2$. In this case, Step 1 in the proof of Proposition 3.5 holds, hence V' is a generalized projective space, a contradiction, since the hyperplane Y of V' is a Bridges space.

Step 2: If $W_V = |V|+1$, then $\mathbb{L}$ is the sum of two Bridges spaces.

From (3.5.1) we have

$$(|V|-|X|)(W_{V'}-W_Y-1) = |V'|+1-W_{V'}. \tag{3.12.1}$$

From the inequalities $|V| \geqslant |X|+1$ and $W_{V'} \geqslant W_Y+1$ we have $W_{V'} \leqslant |V'|+1$, thus $W_{V'} = |V'|+1$, since V' is not a projective space, as it contains the hyperplane Y which is a Bridges space. From (3.12.1) we have $(|V|-|X|)(|V'|-|Y|-1)=0$, hence $|V'|=|Y|+1$ and $V' = x_1 \oplus Y$. In this case $\mathbb{L}$ is the sum of the two Bridges spaces V and Y.

Step 3: If $W_V = |V|$, then either $W_{V'} = |V'|+1$ or $W_{V'} = |V'|+2$.

From (3.5.1) we have

$$(|V|-|X|)(W_{V'}-W_Y-1) = |V'|+1-W_Y. \tag{3.12.2}$$

As $|V| \geqslant |X|+1$, from (3.12.2) it follows $W_{V'} \leqslant |V'|+2$. Since V' contains the Bridges space Y, $W_{V'} = |V'|$ cannot occur, and the statement follows.

In order to complete the proof, we have to examine two subcases:

(i) $W_V = |V|$ and $W_{V'} = |V'|+1$;
(ii) $W_V = |V|$ and $W_{V'} = |V'|+2$.

Case (i): If $W_V = |V|$ and $W_{V'} = |V'|+1$ then $\mathbb{L}$ is the sum of two Bridges spaces.

From (3.12.2) we have $|V|-|X|-1 = |V'|-|Y|-1 = 1$, and then $|V| = |X|+2$ and $|V'| = |Y|+2$. Let x be the unique point of $V \backslash X$ different from x_1. Since V is a projective space, the line $x_1 \vee x$ intersects the hyperplane X of V at a point x' and every line of X passing through x' contains exactly two points, otherwise V does not satisfy the Veblen axiom. It follows that X is the sum of x' and an $(h-1)$-dimensional projective subspace X_1. Moreover, let y be the unique point of $V' \backslash Y$ different from x_1. The line $x_1 \vee y$ intersects B_1 (by the case (iii) of Proposition 3.11) and it is contained in V', thus $x_1 \vee y$ intersects Y at a point y'. Since V' is a Bridges space, the planes of V' are Bridges planes or projective planes. If Y contains a line R passing through y' and containing at least three points, then, necessarily, this line contains either three or four points and the plane $x_1 \vee R$ is either $PG(2,2)$, or $PG(2,2)$ with one point deleted, or $PG(2,3)$ with one point deleted, a contradiction, as $V' \backslash Y$ does not contain any point different from x_1 and y. It follows that Y is the sum of y' and a $(k-1)$-dimensional Bridges space Y_1. Finally, it is clear that the linear space $\mathbb{L}'$ whose point-set is $\{x_1, x, x', y, y'\}$ is the Fano plane with two points deleted, hence, by Theorem 1.2, the linear space $\mathbb{L}' \oplus X_1$ is a Bridges space of dimension $h+2$ and $\mathbb{L}$ is the sum of the two Bridges spaces $\mathbb{L}' \oplus X_1$ and Y_1.

Case (ii): If $W_V = |V|$ and $W_{V'} = |V'|+2$ then $\mathbb{L}$ is the sum of a generalized projective space and a de Witte space.

From (3.12.2) we have $(|V|-|X|-1)(|V'|-|Y|) = 0$, and then $|V| = |X|+1$, as $|V'| \geqslant |Y|+1$. It follows that $V = x_1 \oplus X$ and $\mathbb{L} = X \oplus V'$ is the sum of the generalized projective space X and the de Witte space V'. □

Remark 3.13. By Proposition 3.12, from now on we can suppose that for every $i = 1, \ldots, h$, every hyperplane B_i satisfying the equalities $W_{n-1}(x_i) = s_i = |B_i|+1$ is an $(n-1)$-dimensional projective space $PG(n-1, q_i)$ of order $q_i \geqslant 2$ with one point deleted. This occurrence will be treated in the following subsection.

3.4. The remaining cases: the hyperplanes $B_1, B_2, \ldots, B_h$ either are Galois projective spaces or Galois projective spaces with one point deleted

According to Remarks 3.6 and 3.13, we can suppose that for every $i = 1, \ldots, h$ there exists a prime power $q_i \geqslant 2$ such that either B_i is $PG(n-1, q_i)$ and satisfies $W_{n-1}(x_i) = s_i = |B_i|$, or B_i is $PG(n-1, q_i)$ with one point deleted and satisfies $W_{n-1}(x_i) = s_i = |B_i|+1$. In the following proposition I prove that the order q_i are all equal.

Proposition 3.14. *The hyperplanes $B_1, \ldots, B_h$ pairwise intersect in an $(n-2)$-dimensional subspace. Then, there exists a prime power $q \geqslant 2$ such that every B_i is either $PG(n-1, q)$ or $PG(n-1, q)$ with one point deleted.*

Proof. Let (x_i, B_i) and (x_j, B_j) be two distinct pairs, with $i, j \in \{1, \ldots, h\}$, $i \neq j$. If either x_i is a point of B_j, or x_j is a point of B_i, then the statement follows from Proposition 3.4 and from case (ii) of Proposition 3.11. Hence we can assume that $x_i \notin B_j$ and $x_j \notin B_i$.

First, I prove the following claim.

3.14.1. Any hyperplane of $\mathbb{L}$ passing through x_i (respectively, x_j) intersects B_j (resp. B_i) at an $(n-2)$-dimensional subspace

For every hyperplane C of B_i, $x_j \vee C$ is an hyperplane of $\mathbb{L}$ intersecting B_j at an hyperplane C', and distinct hyperplanes of B_i clearly provide distinct hyperplanes of B_j. It follows that the correspondence mapping every hyperplane C of B_i onto the hyperplane $(x_j \vee C) \cap B_j$ of B_j is injective. Therefore, the number s_i of the hyperplanes of B_i is less than or equal to the number s_j of the hyperplanes of B_j. Obviously, the same arguments work if we interchange B_i and B_j, thus $s_i = s_j$. From Proposition 3.4 and case (ii) of Proposition 3.11, we have that $W_{n-1}(x_i) = s_i = s_j = W_{n-1}(x_j)$, and claim (3.14.1) is proved.

If B_i is a $PG(n-1, q_i)$ with one point deleted, then, adding a point, say ∞, to every line of size q_i of B_i, the linear space whose point-set is $\widehat{B_i} = B_i \cup \{\infty\}$ is precisely $PG(n-1, q_i)$. By case (iii) of Proposition 3.11, it is clear that if the line $x_i \vee x_j$ does not intersects the hyperplane B_i, then $x_i \vee x_j$ contains the point ∞ of $\widehat{B_i}$.

Suppose now, by contradiction, that $B_i \cap B_j$ is a subspace of dimension at most $n-3$. Let x be either the point $(x_i \vee x_j) \cap B_i$, in the case $B_i = PG(n-1, q_i)$, or the point ∞, in the case $\widehat{B_i} = PG(n-1, q_i)$, and consider an hyperplane C of B_i not containing x and $B_i \cap B_j$. The hyperplanes $x_i \vee C$ and $x_j \vee C$ of $\mathbb{L}$ are distinct (otherwise x is a point of C) and, from Proposition 3.4, case (ii) of Proposition 3.11 and condition (3.14.1) above, $(x_i \vee C) \cap B_j = C'$ and $(x_j \vee C) \cap B_j = C''$ are two hyperplanes of B_j. Since $C' \cap C'' = (x_i \vee C) \cap (x_j \vee C) \cap B_j = C \cap B_j$, $C' \cap C''$ has dimension at most $n-4$, a contradiction, since C' and C'' are hyperplanes either of a projective space or of a projective space with one point deleted, hence their intersection is an $(n-3)$-subspace. It follows that for every $i, j = 1, \ldots, h$, $q_i = q_j = q$. $\square$

Let k be a non-negative integer $0 \leqslant k \leqslant h$ such that the first k hyperplanes B_i for $i = 1, \ldots, h$ are $PG(n-1, q)$ and the last $h-k$ are $PG(n-1, q)$ with one point deleted. I explicitly observe that, if B_i is $PG(n-1, q)$ with one point deleted, then, adding a point, say ∞_i, to every line of B_i containing exactly q points, the linear space whose point-set is $\widehat{B_i} = B_i \cup \{\infty_i\}$ is precisely $PG(n-1, q)$. In the sequel, the symbol B_α (with the greek letter) will denote either B_i or $\widehat{B_i}$, according to B_i being a projective space or a projective space with one point deleted.

From Proposition 3.4, for every $j \leqslant k$ the residue $\mathbb{L}_{x_j}$ of $\mathbb{L}$ at x_j is isomorphic to B_j, hence it is a projective space $PG(n-1, q)$. Moreover, from case (iii) of Proposition 3.11, for every $j = k+1, \ldots, h$, the residue $\mathbb{L}_{x_j}$ of $\mathbb{L}$ at the point x_j is a projective space $PG(n-1, q)$, isomorphic to $\widehat{B_j}$. In this case, the unique line passing through x_j and missing B_j clearly intersects $\widehat{B_j}$ at the point ∞_j.

By arguments similar to those of [8], the following results hold.

Proposition 3.15. (i) *Every line of $\mathbb{L}$ contains at most $q+1$ points.*

(ii) *$\mathbb{L}$ contains at most $q^n + q^{n-1} + \cdots + q + 1$ points.*

(iii) *For every $i = 1, \ldots, h$, the number β_i of the hyperplanes of $\mathbb{L}$ intersecting B_i at an $(n-2)$-subspace is at least $q^n + q^{n-1} + \cdots + q^2 + 1$.*

(iv) *For every $j = 1, \ldots, n-1$ and for every $\alpha = 1, \ldots, h$, every j-subspace X of $\mathbb{L}$ not contained in B_α intersects B_α at a $(j-1)$-dimensional subspace.*

(v) *Every hyperplane of $\mathbb{L}$ contains at most $q^{n-1} + q^{n-2} + \cdots + q + 1$ points.*

Notice that in the previous proof some care is needed in the fact that, for every $t = 1, \ldots, n-1$, every t-dimensional subspace of B_i either contains $\vartheta_t = (q^{t+1}-1)/(q-1)$ points, or $\vartheta_t - 1$ points, according to B_i either is $PG(n-1, q)$ or $PG(n-1, q)$ with one point deleted.

Recall that, as a consequence of Proposition 3.1, we have labelled the hyperplanes of $\mathbb{L}$ in such a way that the complements of $B_1, \ldots, B_v$ have a transversal $\{x_1, \ldots, x_v\}$ and, by Proposition 3.3, the first $h \geqslant 2$ of them satisfy the condition $|B_i| \leqslant W_{n-1}(x_i) \leqslant |B_i| + 1$. We have the following properties.

Proposition 3.16. (i) $h \geqslant q^n - q - 1$.

(ii) *The subspace $\bigcap_{i=1}^{h} B_i$ has dimension at most $n-3$.*

(iii) *For every* $i=1,\dots,h$, *every hyperplane of* B_i *is contained in exactly* q *hyperplanes of* $\mathbb{L}$ *different from* B_i.
(iv) $W_{n-1}=q^n+q^{n-1}+\cdots+q+1$.

Proposition 3.17. *The subspace* $\bigcap_{i=1}^h B_i$ *is empty.*

Proof. Let k be the number of distinct hyperplanes $B_i\cap B_j$ of B_i such that the subspace

$$S=\bigcap_{i=1}^{h} B_i=\bigcap_{\substack{j=1\\ j\neq i}}^{h}(B_i\cap B_j)$$

has dimension $n-1-k$, with $2\leqslant k\leqslant h-1$. From (i) and (iii) of Proposition 3.16, we have $q^n-q-1\leqslant h\leqslant kq+1$, then $k\geqslant (q^n-q-2)/q=q^{n-1}-1-2/q$, and hence $k\geqslant q^{n-1}-2$, since $q\geqslant 2$. If $n-1-k\geqslant 0$, then $n+1\geqslant q^{n-1}$, a contradiction, since $n\geqslant 4$ and $q\geqslant 2$. Thus, S has dimension -1 and the hyperplanes B_i intersect at the empty set. □

Finally, we can conclude with the following result.

Proposition 3.18. *If the hyperplanes* $B_1, B_2, \dots, B_h$ *are either Galois projective spaces or Galois projective spaces with one point deleted, then the linear space* $\mathbb{L}$ *is a projective space* $PG(n,q)$ *with two points deleted.*

Proof. From (iv) of Proposition 3.16, it follows $v=W_{n-1}-2=q^n+\cdots+q-1$. Let x be a fixed point of $\mathbb{L}$. Then there exists an index $i=1,\dots,h$ such that the hyperplane B_i of $\mathbb{L}$ does not contain x (from Proposition 3.17). As before, let B_α be either the hyperplane B_i, in the case B_i is a $PG(n-1,q)$, or $\widehat{B_i}=B_i\cup\{\infty_i\}$, if B_i is $PG(n-1,q)$ with one point deleted. Moreover, by (iv) and (i) of Proposition 3.15, every line through x intersects B_α and contains at most $q+1$ points. Let ζ be the number of lines passing through x and containing at most q points. Since $\mathscr{P}=\bigcup_{y\in B_\alpha}(x\vee y)$, we have $q^n+\cdots+q-1=v\leqslant(q^{n-1}+\cdots+q+1-\zeta)q+\zeta(q-1)+1=q^n+\cdots+q^2+q-\zeta+1$, i.e. $\zeta\leqslant 2$.

Let $\zeta=0$: Then every line through x contains exactly $q+1$ points. Hence either $v=q(q^{n-1}+\cdots+q+1)+1$, in the case $B_i=PG(n-1,q)$, or $v=q(q^{n-1}+\cdots+q)+(q-1)+1$, if B_i is $PG(n-1,q)$ with one point deleted. In both cases, we have a contradiction.

Let $\zeta=1$: Then there exists a unique line R passing through x and containing $k\leqslant q$ points. We have the following possibilities:

(i) $v=q(q^{n-1}+\cdots+q)+(k-1)+1$, if $B_i=PG(n-1,q)$; in this case, $k=q-1$.
(ii) $v=q(q^{n-1}+\cdots+q)+(k-2)+1$, if B_i is $PG(n-1,q)$ with one point deleted and the line R contains ∞_i; in this case, $k=q$.
(iii) $v=q(q^{n-1}+\cdots+q-1)+(q-1)+(k-1)+1$, if B_i is $PG(n-1,q)$ with one point deleted and R does not contain ∞_i; in this case, $k=q$.

In a very natural way, in the case (i) we can add two "virtual" points a and b to the line R. In fact, let us suppose that every line of $\mathbb{L}$ disjoint from R and spanning with R a 2-subspace contains either a or b. Moreover, if L, M, and R are contained into the same 2-subspace π and they are pairwise disjoint, then L and M contain the same virtual point, otherwise, if $L\cap R=M\cap R=\emptyset$ and $L\cap M\neq\emptyset$, then L and M contain different virtual points. Finally, if L and M are two lines of $\mathbb{L}$ disjoint from R and spanning with R two different two-subspaces, then L and M contain the same virtual point if, and only if, they span a two-subspace. By (iv) of Proposition 3.16 and by Theorem 1.1, the linear space $\mathbb{L}'$ whose point-set is $\mathscr{P}\cup\{a,b\}$ is a projective space $PG(n,q)$.

Analogously, if we add a "virtual" point c to the line R even in case (ii) and (iii), the linear space $\mathbb{L}''$ whose point-set is $\mathscr{P}\cup\{c,\infty_i\}$ is a projective space $PG(n,q)$.

Let $\zeta=2$: Then there exist exactly two lines R_1 and R_2 passing through x and containing k_1 and k_2 points, respectively, with $k_1,k_2\leqslant q$. We have the following possibilities:

(iv) $v=q(q^{n-1}+\cdots+q-1)+(k_1-1)+(k_2-1)+1$, if $B_i=PG(n-1,q)$; in this case $k_1+k_2=2q$, thus $k_1=k_2=q$.

(v) $v = q(q^{n-1} + \cdots + q - 1) + (k_1 - 1) + (k_2 - 2) + 1$, if B_i is $PG(n-1, q)$ with one point deleted and one of the lines R_1 and R_2, suppose R_2, contains ∞_i; in this case $k_1 + k_2 = 2q + 1$, a contradiction, since $k_1, k_2 \leqslant q$.

(vi) $v = q(q^{n-1} + \cdots + q - 2) + (q - 1) + (k_1 - 1) + (k_2 - 1) + 1$ if B_i is $PG(n-1, q)$ with one point deleted and R_1 and R_2 do not contain ∞_i; in this case $k_1 + k_2 = 2q + 1$, a contradiction, since $k_1, k_2 \leqslant q$.

It follows that only case (iv) occurs when $\zeta = 2$. In this case, adding two "virtual" points a_1 to the line R_1 and a_2 to the line R_2, by (iv) of Proposition 3.16 and by Theorem 1.1, the linear space $\mathbb{L}'$ whose point-set is $\mathscr{P} \cup \{a_1, a_2\}$ is a projective space $PG(n, q)$. $\square$

Proof of Theorem 1.4. Let $\mathbb{L}$ be a de Witte space of dimension $n \geqslant 4$ containing v points. If $\mathbb{L}$ contains $v+1$ hyperplanes with a non-empty intersection $\mathscr{I}$, then, by Theorem 1.5 for $\alpha = 2$, $\mathscr{I}$ is a point x_0, the remaining hyperplane B_0 is an $(n-1)$-dimensional de Witte space on $v - 1$ points and $\mathbb{L}$ is the sum $x_0 \oplus B_0$. Suppose now that $v + 1$ hyperplanes of $\mathbb{L}$ always intersect in the empty set. By Propositions 3.1 and 3.3, there exist an index $h \geqslant 2$, h hyperplanes $B_1, \ldots, B_h$ and h points $x_1, \ldots, x_h$ such that, for every $i \in \{1, \ldots, h\}$, $x_i \notin B_i$ and either $|B_i| = s_i = W_{n-1}(x_i)$, or $|B_i| = s_i$, $W_{n-1}(x_i) = |B_i| + 1$, or $s_i = W_{n-1}(x_i) = |B_i| + 1$. By Propositions 3.5, 3.10 and 3.12, Remarks 3.6 and 3.13 and by Proposition 3.18, the proof of Theorem 1.4 follows. $\square$

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